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SCALAR CURVATURES ON S^2

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ABSTRACT. A theorem for the existence of solutions of the nonlinear elliptic equation $-\Delta u + 2 = R(x)e^u$, $x \in S^2$, is proved by using a "mass center" analysis technique and by applying a continuous "flow" in $H^1(S^2)$ controlled by ∇R.

0. Introduction. Given a function $R(x)$ on the two dimensional unit sphere S^2, one wishes to know when it can actually be the scalar curvature of some metric g that is pointwise conformal to the standard metric g_0 on S^2. This is an interesting problem in geometry (cf. [**1**]). In order to find an answer, people usually consider the differential equation

$$(*) \qquad \Delta u - 2 + R(x)e^u = 0, \qquad x \in S^2.$$

It is well known that if u is a solution of $(*)$, then $R(x)$ turns out to be the scalar curvature corresponding to the metric $g = e^u g_0$, which, obviously is pointwise conformal to g_0.

There are some necessary conditions for the solvability of $(*)$ pointed out by Kazdan and Warner (cf. [**2**]), which show that not all smooth functions $R(x)$ can be achieved as such a scalar curvature. Then for which R can one solve $(*)$? This has been an open problem for many years (cf. [**3**]).

Moser [**4**] proved that if $R(x) = R(-x)$, for any $x \in S^2$, and R is positive somewhere, then $(*)$ has a solution. Recently, Hong [**5**] considered the case where R is rotationally symmetric and established some existence theorems. In our previous paper [**6**], we generalized the results of Moser and Hong to the case where R possesses some kinds of generic symmetries, that is, R is invariant under the action of some subgroups of the orthogonal transformation group in $\mathbf{R}^3$. Then it is natural for one to ask, "What happens when R is not symmetric?" So far we know, there have not yet been any existence results in this situation. This is the motivation for this present paper.

In this paper, without any symmetry assumption on R, we find some sufficient conditions so that $(*)$ can be solved, which is independent of the results in [**6**]. To find a solution of $(*)$, we consider the functional

$$J(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2\,dA + 2\int_{S^2} u\,dA - 8\pi \ln \int_{S^2} Re^u\,dA$$

defined on

$$H_* = \left\{ u \in H^1(S^2)\colon \int_{S^2} Re^u\,dA > 0 \right\}$$

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and seek critical points of J. It is easily seen that a critical point of J in H_* plus a suitable constant makes a solution of $(*)$. Since J is bounded from below on H_*, a natural idea is to seek a minimum of J. Unfortunately, it was shown (cf. [**5**]) that $\inf_{H_*} J$ can never be attained unless R is a constant. So one is led to find saddle points of J. Under some appropriate assumptions on R, using a family of transformations on $H^1(S^2)$ and a continuous "flow" in H_*, by a careful "mass center" analysis, we prove the existence of a saddle point of J in H_* and establish the following

THEOREM. *Assume*
(R_0) $R \in C^2(S^2)$.
(R_1) *There exist two points, say a and b, on S^2, such that*

$$R(a) = R(b) = m \equiv \max_{S^2} R > 0,$$

and

$$\sup_{h\in\Gamma} \min_{x\in h([0,1])} R(x) = \nu < m,$$

where $\Gamma = \{h\colon h \in C([0,1], S^2),\ h(0) = a,\ h(1) = b\}$.
(R_2) *There is $h_0 \in \Gamma$ such that $\min_{h_0([0,1])} R = \nu$, and for any*

$$x \in K \equiv \{x \in h_0([0,1])\colon R(x) = \nu\}, \qquad \Delta R(x) > 0.$$

(R_3) *There is no critical value of R in the interval (ν, m).*
Then problem $(*)$ possesses at least one solution.

OUTLINE OF THE PROOF. Due to its complexity, we divide our proof into five sections.

In §1, we find two families of separated points, say $\{\varphi_{\lambda,a}\}$ and $\{\varphi_{\lambda,b}\}$ with $\lambda \in [0,1)$, satisfying

$$J(\varphi_{\lambda,a}), J(\varphi_{\lambda,b}) \to \inf_{H_*} J, \quad \text{as } \lambda \to 1.$$

And under the condition (R_1) prove that as λ gets sufficiently close to 1, there exists a "mountain pass" between the two points $\varphi_{\lambda,a}$ and $\varphi_{\lambda,b}$, i.e.

$$\mu_\lambda = \inf_{l\in L_\lambda} \max_{u\in l([0,1])} J(u) > \max\{J(\varphi_{\lambda,a}), J(\varphi_{\lambda,b})\}. \tag{0.1}$$

where $L_\lambda = \{l\colon l \in C([0,1], H_*),\ l(0) = \varphi_{\lambda,a}, l(1) = \varphi_{\lambda,b}\}$. Now, by Ekeland's variational principle (see [**7**]) there is a sequence $\{u_k\}$ in H_* such that $J(u_k) \to \mu_\lambda$, $J'(u_k) \to 0$, as $k \to \infty$. If $\{u_k\}$ possesses a strongly convergent subsequence, then we have solved our problem.

When does the sequence $\{u_k\}$ converge strongly? In order to investigate this, we verify, in §2 a modified (P.S.) condition for the functional J, that is

PROPOSITION 2.1. *Assume $\{v_k\} \subset H_*$, $J(v_k) \le \beta < +\infty$, $J'(v_k) \to 0$, as $k \to \infty$, and $|P(v_k)| \le 1 - \gamma < 1$. Let $\tilde{v}_k = v_k - (1/4\pi)\int_{S^2} v_k\, dA$. Then $\{\tilde{v}_k\}$ possesses a strongly convergent subsequence in H_*, whose limit v_0 verifies $J'(v_0) = 0$ where $P(u)$ stands for the mass center of the function $e^{u(x)}$ defined on S^2.*

Due to Proposition 2.1, the key point to the solution of problem $(*)$ lies in controlling the behavior of the sequence $\{P(u_k)\}$. To do this, we divide, according to the value of R, the sphere S^2 into several areas, and try to find such a sequence $\{u_k\}$ that $\{P(u_k)\}$ can reach none of the areas on the sphere.

Thus, we introduce in §3 a family of transformations on $H^1(S^2)$ which leaves the functionals

$$F(u) = \frac{1}{2}\int_{S^2} |\nabla u|^2 \, dA + 2\int_{S^2} u \, dA \quad \text{and} \quad G(u) = \int_{S^2} e^u \, dA$$

invariant. With the help of this family of transformations, we obtain in §4 the following propositions providing some useful information when $P(u_k) \to S^2$.

PROPOSITION 4.2. *Suppose $\{v_k\} \subset H_*$, $\{J(v_k)\}$ is bounded; $J'(v_k) \to 0$, and $P(v_k) \to \varsigma \in S^2$, as $k \to \infty$. Then there exists a subsequence $\{v_{k_i}\}$ of $\{v_k\}$, and α_i, ς_i, with $\alpha_i \to 1$, $\varsigma_i \to \varsigma$ as $i \to \infty$ such that $\int_{S^2} |\nabla(v_{k_i} - \varphi_{\alpha_i,\varsigma_i})|^2 \to 0$ as $i \to \infty$, where $\varphi_{\lambda,\varsigma}(x) = \ln[(1-\lambda^2)/(1-\lambda\cos r(x,\varsigma))^2]$, with $r(x,\varsigma)$ the geodesic distance between the two points x and ς on S^2.*

PROPOSITION 4.3. *Let $\{v_k\} \subset H_*$, $J(v_k)$ bounded, and $J'(v_k) \to 0$, $P(v_k) \to \varsigma \in S^2$ with $R(\varsigma) > 0$. Then there is a subsequence $\{v_{k_i}\}$ of $\{v_k\}$ such that $J(v_{k_i}) \to 8\pi \ln 4\pi R(\varsigma)$.*

PROPOSITION 4.4. *Assume $\{u_k\}$, $\{v_k\}$ in H_* satisfying*
(1) $\{J(u_k)\}$, $\{J(v_k)\}$ are bounded; and $J'(v_k) \to 0$, as $k \to \infty$.
(2) $\int_{S^2} |\nabla(u_k - v_k)|^2 \, dA \to 0$, as $k \to \infty$.
(3) $P(u_k) \to \eta \in S^2$, $P(v_k) \to \varsigma \in S^2$, as $k \to \infty$.
Then $\eta = \varsigma$.

Condition (R_2) enables us to establish the estimate

$$\mu_\lambda < -8\pi \ln 4\pi\nu$$

for λ sufficiently close to 1. Then for such λ we prove

PROPOSITION 4.6. *There exist α_0, $\delta_0 > 0$, such that for any $\{v_k\}$ in H_*, if $J(v_k) \le \mu_\lambda + \delta_0$ $(k = 1, 2, \dots)$ and $P(v_k) \to \varsigma \in S^2$, as $k \to \infty$ then $R(\varsigma) \ge \nu + \alpha_0$.*

Finally, in §5, we utilize ∇R to construct a continuous "flow" in H_*. Based on the results in the proceding sections, mainly in §4, and applying the "flow", we are able to pick a sequence $\{u_k\}$ in H_*, such that as $k \to \infty$, $J(u_k) \to \mu_{\lambda_0}$ (for some λ_0 sufficiently close to 1), $J'(u_k) \to 0$, and $\{P(u_k)\}$ is bounded away from the sphere S^2. Therefore we arrive at the conclusion that μ_{λ_0} is a critical value of J, and complete the proof of our Theorem.

REMARK 0.1. In the Theorem, if $\nu \le 0$, then assumption (R_2) can be omitted.

REMARK 0.2. Condition (R_1) may be generalized as

$(\tilde{R}_1)$ Let $m = \max_{S^2} R$, $M = R^{-1}(m)$; M is not contractible in itself, but is contractible on S^2. Let

$$\Gamma = \left\{ U = \bigcup_{t\in[0,1]} h_t(M) \;\middle|\; \begin{array}{l} h_t(\cdot) \text{ is a deformation of } M \text{ on } S^2; \\ h_0(M) = M \text{ and } h_1(M) \text{ is a point on } S^2 \end{array} \right\}$$

and suppose that $\nu = \sup_{U\in\Gamma} \min_{x\in U} R(x) < m$. Then by a similar argument as in the proof of our Theorem, one can show that condition (R_0), $(\tilde{R}_1)$, (R_2) and (R_3) are sufficient for problem $(*)$ to have a solution.

We assume (R_0)–(R_3) throughout the paper.

1. Mountain pass. Consider the functional J defined on H_*. For x, $\varsigma \in S^2$, $\lambda \in [0,1)$, define

$$\varphi_{\lambda,\varsigma}(x) = \ln \frac{1-\lambda^2}{(1-\lambda\cos r(x,\varsigma))^2}$$

where $r(x,\varsigma)$ is the geodesic distance between two points x and ς on S^2. A direct computation shows that, as $\lambda \to 1$,

$$J(\varphi_{\lambda,a}) = -8\pi\ln\int_{S^2} Re^{\varphi_{\lambda,a}}\,dA \to -8\pi\ln 4\pi R(a) = -8\pi\ln 4\pi m.$$

Similarly,

$$J(\varphi_{\lambda,b}) \to -8\pi\ln 4\pi m. \tag{1.1}$$

On the other hand, the inequality (cf. [**5**])

$$\int_{S^2} e^u\,dA \le 4\pi\exp\left(\frac{1}{16\pi}\int_{S^2}|\nabla u|^2\,dA + \frac{1}{4\pi}\int_{S^2} u\,dA\right) \quad \forall u \in H^1(S^2), \tag{1.2}$$

leads to

$$J(u) \ge -8\pi\ln 4\pi m \quad \forall u \in H_*.$$

Therefore,

$$\inf_{H_*} J = -8\pi\ln 4\pi m. \tag{1.3}$$

(1.1) and (1.3) inform us that there are two separated points, say $\varphi_{\lambda,a}$ and $\varphi_{\lambda,b}$, in H_* at which the values of J are as close to $\inf_{H_*} J$ as we wish. This phenomenon would naturally lead one to expect that there might be a "mountain pass" between the two separated points. In order to show this, we need the concept of mass center introduced in [**6**]. Now let us first recall it.

For $u \in H^1(S^2)$, regard S^2 as a rigid body with density $e^{u(x)}$ at point $x \in S^2$, denote the mass center of this rigid body by $P(u)$. Then in an orthogonal coordinate system $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$,

$$P(u) = \left(\frac{\int_{S^2} x_1 e^u\,dA}{\int_{S^2} e^u\,dA}, \frac{\int_{S^2} x_2 e^u\,dA}{\int_{S^2} e^u\,dA}, \frac{\int_{S^2} x_3 e^u\,dA}{\int_{S^2} e^u\,dA}\right).$$

This concept and its analysis played an important role in [**6**] and will be a powerful tool in the following investigation.

Define $Q(u) = P(u)/|P(u)|$, and $d(u) = |Q(u) - P(u)|$, $u \in H^1(S^2)$.

LEMMA 1.1. *There exists constant* $C_0 = C(|R|_{C^1(S^2)})$, *such that*

$$\left|\int_{S^2}\{R(x) - R(Q(u))\}e^u\,dA\right| \le C_0\sqrt[3]{d(u)}\int_{S^2} e^u\,dA. \tag{1.4}$$

PROOF. For simplicity, write $Q = Q(u) = (Q_1, Q_2, Q_3)$, and $S_r = S_r(Q) = \{x \in S^2 \colon r(x,Q) < r\}$. Choose $r = \sqrt[3]{d(u)}$. Since

$$\int_{S^2\setminus S_r}\left(1 - \sum_i x_i Q_i\right)e^u\,dA \Big/ \int_{S^2} e^u\,dA \le 1 - |P(u)| = d(u)$$

and

$$1 - \sum_i x_i Q_i \ge r^2/4 \quad \forall x \in S^2\setminus S_r,$$

we have

$$\int_{S^2\setminus S_r} e^u\,dA \Big/ \int_{S^2} e^u\,dA \le 4\sqrt[3]{d(u)}.$$

Therefore,

$$\left|\int_{S^2}\{R(x)-R(Q)\}e^u\,dA\right| \le \left|\int_{S_r}\{R(x)-R(Q)\}e^u\,dA\right| + \left|\int_{S^2\setminus S_r}\{R(x)-R(Q)\}e^u\right|$$
$$\le \left\{\max_{S^2}|\nabla R|\cdot r + 8\max_{S^2}|R|\sqrt[3]{d(u)}\right\}\int_{S^2} e^u\,dA \le C_0\sqrt[3]{d(u)}\int_{S^2} e^u\,dA.$$

LEMMA 1.2. *Let $u\in H_*$, $J(u)\le\beta$, $|P(u)|\le 1-\gamma<1$. Then*

$$\int_{S^2}|\nabla u|^2\,dA \le C(\beta,\gamma).$$

PROOF. Analogous to the proof of Proposition 1.2 in [**6**].

Define

$$L_\lambda = \{l\colon l\in C([0,1],H_*), l(0)=\varphi_{\lambda,a}, l(1)=\varphi_{\lambda,b}\},$$
$$\mu_\lambda = \inf_{l\in L_\lambda}\max_{u\in l([0,1])} J(u).$$

For λ sufficiently close to 1, L_λ is nonempty. In fact, since $R(a)=R(b)>0$, one has $\int_{S^2} Re^{\varphi_{\lambda,a}}\,dA$, $\int_{S^2} Re^{\varphi_{\lambda,b}}\,d\Lambda>0$ for λ close to 1. Take

$$u_\lambda^t = \ln[(1-t)e^{\varphi_{\lambda,a}} + te^{\varphi_{\lambda,b}}], \qquad t\in[0,1];$$

then $\int_{S^2} Re^{u_\lambda^t}\,dA>0$; hence $l_\lambda=\{u_\lambda^t : t\in[0,1]\}\in L_\lambda$.

PROPOSITION 1.3 (MOUNTAIN PASS). *For λ sufficiently close to* 1,

$$\mu_\lambda > \max\{J(\varphi_{\lambda,a}), J(\varphi_{\lambda,b})\}. \tag{1.5}$$

PROOF. We argue indirectly. Suppose there exists $\{\lambda_k\}$, $\lambda_k\to 1$, such that

$$\mu_{\lambda_k}\le \max\{J(\varphi_{\lambda_k,a}), J(\varphi_{\lambda_k,b})\}, \qquad k=1,2,\dots.$$

Then by (1.1), one can find $\{\varepsilon_k\}$, $\varepsilon_k\to 0$, and $\mu_{\lambda_k}<-8\pi\ln 4\pi m+\varepsilon_k$. By the definition of μ_k, there is $l_k\in L_{\lambda_k}$ such that

$$\max_{l_k([0,1])} J < -8\pi\ln 4\pi m+\varepsilon_k. \tag{1.6}$$

Let $d_0>0$ be sufficiently small, so that if $d(u)\le d_0$, then

$$C_0\sqrt[3]{d(u)} < \tfrac{1}{2}(m-\nu). \tag{1.7}$$

By (R_1), this is possible.

Case (1). There exists k_0 such that if $k\ge k_0$, then for any u on l_k, $d(u)\le d_0$. Then by (1.6) and (R_1), one can pick some $k\ge k_0$ so that

$$\max_{l_k([0,1])} J < -8\pi\ln 2\pi(m+\nu). \tag{1.8}$$

Fix such k and set $h(t)=Q(l_k(t))$, $t\in[0,1]$. It is obvious that $h([0,1])$ is a continuous curve on S^2 joining a and b. Let $t_0\in(0,1)$ satisfy

$$R(h(t_0)) = \min_{h([0,1])} R.$$

Then

$$R(h(t_0)) \leq \nu. \tag{1.9}$$

Write $v = l_k(t_0)$, $Q = Q(v) = h(t_0)$. Then by (1.4) and (1.7)–(1.9),

$$\begin{aligned} -8\pi \ln 2\pi(m+\nu) &> J(v) \\ &\geq \frac{1}{2}\int_{S^2} |\nabla v|^2\, dA + 2\int_{S^2} v\, dA - 8\pi\ln \int_{S^2} e^v\, dA - 8\pi\ln(R(Q) + C_0 \sqrt[3]{d(v)}) \\ &\geq -8\pi\ln 4\pi - 8\pi\ln\tfrac{1}{2}(\nu+m) = -8\pi\ln 2\pi(m+\nu), \end{aligned}$$

a contradiction.

Case (2). There exists a subsequence of $\{l_k\}$ (still denoted by $\{l_k\}$) such that for any k, one can pick out a $u_k \in l_k$ satisfying $d(u_k) > d_0$. Then by Lemma 1.2, we infer that $\int_{S^2} |\nabla u_k|^2\, dA$ are bounded, which implies that the sequence $\{\tilde{u}_k = u_k - (1/4\pi)\int_{S^2} u_k\, dA\}$ is bounded in $H^1(S^2)$ and hence possesses a subsequence coverging weakly to an element, say u_0, in $H^1(S^2)$. Since $u_k \in H_*$, it is easy to verify that $\tilde{u}_k$ and the weak limit u_0 are in H_*; consequently, $J(u_0) = \inf_{H_*} J$. This is impossible because by (R_1) R is not constant, so $\inf_{H_*} J$ can never be attained.

The above argument shows that our hypothesis at the beginning of the proof is false, so (1.5) must hold for λ sufficiently close to 1.

2. A modified (P.S.) condition.

PROPOSITION 2.1. *Assume $\{u_k\} \subset H_*$, $J(u_k) \leq \beta < +\infty$, $J'(u_k) \to 0$, as $k \to \infty$, and also assume $|P(u_k)| \leq 1 - \gamma < 1$. Let $\tilde{u}_k = u_k - (1/4\pi)\int_{S^2} u_k\, dA$. Then $\{u_k\}$ possesses a strongly convergent subsequence in $H^1(S^2)$ whose limit u_0 verifies $J'(u_0) = 0$.*

PROOF. In the proof of Proposition 1.3, we have already seen that $\{\tilde{u}_k\} \subset H_*$ and there is a subsequence of $\{\tilde{u}_k\}$ (still denoted by $\{\tilde{u}_k\}$) converging weakly to u_0 in H_*.

Since for any constant C, $J(u+C) = J(u)$, one concludes, from the definition of J', that

$$J'(\tilde{u}_k) = J'(u_k) \to 0, \quad \text{as } k \to \infty. \tag{2.1}$$

Hence

$$-\Delta \tilde{u}_k - \frac{8\pi}{\int_{S^2} Re^{\tilde{u}_k}\, dA} Re^{\tilde{u}_k} = 2 + o(1).$$

Consequently,

(2.2)

$$\begin{aligned} \int_{S^2} |\nabla(\tilde{u}_i - \tilde{u}_j)|^2 &= 8\pi \int_{S^2} \left\{ \frac{Re^{\tilde{u}_i}}{\int_{S^2} Re^{\tilde{u}_i}} - \frac{Re^{\tilde{u}_j}}{\int_{S^2} Re^{\tilde{u}_j}} \right\} (\tilde{u}_i - \tilde{u}_j)\, dA + o(1) \\ &\leq 8\pi \left(\int_{S^2} \left| \frac{Re^{\tilde{u}_i}}{\int_{S^2} Re^{\tilde{u}_i}} - \frac{Re^{\tilde{u}_j}}{\int_{S^2} Re^{\tilde{u}_j}} \right|^2 dA \right)^{1/2} \left(\int_{S^2} (\tilde{u}_i - \tilde{u}_j)^2\, dA \right)^{1/2} + o(1). \end{aligned}$$

The boundedness of $\{\tilde{u}_k\}$ (in $H^1(S^2)$) and of $J(\tilde{u}_k)$ leads to

$$\int_{S^2} Re^{\tilde{u}_k}\, dA \geq \alpha > 0, \qquad k = 1, 2, \ldots,$$

which, together with (1.2), implies that, for any $i, j = 1, 2, \dots$, the integrals

$$\int_{S^2} \left\{ \frac{Re^{\tilde{u}_i}}{\int_{S^2} Re^{\tilde{u}_i}} - \frac{Re^{\tilde{u}_j}}{\int_{S^2} Re^{\tilde{u}_j}} \right\}^2 dA$$

are bounded. By the compact embedding $H^1(S^2) \hookrightarrow L^2(S^2)$, we have

$$\int_{S^2} (\tilde{u}_i - \tilde{u}_j)^2 \, dA \to 0, \quad \text{as } i, j \to \infty.$$

Now, it follows from (2.2) that

$$\int_{S^2} |\nabla(\tilde{u}_i - \tilde{u}_j)|^2 \, dA \to 0, \quad \text{as } i, j \to \infty.$$

Consequently,

$$\tilde{u}_k \to u_0 \quad \text{in } H^1(S^2).$$

Therefore, by (2.1), $J'(u_0) = 0$. This completes the proof.

Let λ_0 be so close to 1 that (1.5) is valid. Then by Ekeland's variational principle (cf. [**7**], the proof for Mountain Pass Lemma), there is a sequence $\{u_k\} \subset H_*$ satisfying $J(u_k) \to \mu_{\lambda_0}$ and $J'(u_k) \to 0$ as $k \to \infty$. Set $\tilde{u}_k = u_k - (1/4\pi) \int_{S^2} u_k \, dA$; then from Proposition 2.1, we know that whether $\{\tilde{u}_k\}$ converges strongly in $H^1(S^2)$ depends on the behavior of the mass center $P(u_k)$. Does $\{P(u_k)\}$ remain bounded away from the sphere S^2? This has now become a key point to the solution of $(*)$. In order to analyze this, we introduce in the following section a family of transformations on $H^1(S^2)$ which possesses some important properties and is very useful in our later investigations.

3. A family of transformations on $H^1(S^2)$. Let $u \in H^1(S^2)$, $\varsigma \in S^2$. Select a spherical polar coordinate system $x = (\theta, \varphi)$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$, so that $\varsigma = (0, \varphi)$. Define a family of transformations $A_{\lambda,\varsigma}$ by

$$A_{\lambda,\varsigma} u(\theta, \varphi) = u \circ h_{\lambda,\varsigma}(\theta, \varphi) + \psi_{\lambda,\varsigma}(\theta)$$

where $0 < \lambda \le 1$, $h_{\lambda,\varsigma}(\theta, \varphi) = (2 \tan^{-1}(\lambda \tan(\theta/2)), \varphi)$ is a conformal transformation on S^2, and

$$\psi_{\lambda,\varsigma}(\theta) = \ln \frac{\lambda^2}{(\cos^2(\theta/2) + \lambda^2 \sin^2(\theta/2))^2}.$$

The following propositions describe some important properties of $A_{\lambda,\varsigma}$.

PROPOSITION 3.1. *Define*

$$I(u) = \frac{1}{2} \int_{S^2} |\nabla u|^2 \, dA + 2 \int_{S^2} u \, dA - 8\pi \ln \int_{S^2} e^u$$

for $u \in H^1(S^2)$. Then

$$I(A_{\lambda,\varsigma} u) = I(u), \qquad \lambda \in (0, 1], \ \varsigma \in S^2; \tag{3.1}$$

and consequently, if u is a solution of the equation

$$-\Delta u + 2 = 2e^u \tag{$**$}$$

then $A_{\lambda,\varsigma} u$ is also a solution.

PROOF. (1)

$$\int_{S^2} e^{A_{\lambda,\varsigma} u} \, dA = \int_0^{2\pi} \int_0^{\pi} e^{u(2 \tan^{-1}(\lambda \tan(\theta/2)), \varphi)} e^{\psi_{\lambda,\varsigma}(\theta)} \sin\theta \, d\theta \, d\varphi.$$

Let $\theta' = 2\tan^{-1}(\lambda\tan(\theta/2))$, $\varphi' = \varphi$. Then $e^{\psi_{\lambda,\varsigma}(\theta)}\sin\theta\, d\theta = \sin\theta'\, d\theta'$. Hence

$$\int_{S^2} e^{A_{\lambda,\varsigma}u}\, dA = \int_0^{2\pi}\int_0^{\pi} e^{u(\theta',\varphi')}\sin\theta'\, d\theta'\, d\varphi' = \int_{S^2} e^u\, dA. \tag{3.2}$$

(2)

(3.3)

$$\begin{aligned}\int_{S^2}|\nabla(A_{\lambda,\varsigma}u)|^2\, dA &= \int_{S^2}|\nabla(u\circ h_{\lambda,\varsigma})|^2\, dA - 2\int_{S^2} u\circ h_{\lambda,\varsigma}\Delta\psi_{\lambda,\varsigma} + \int_{S^2}|\nabla\psi_{\lambda,\varsigma}|^2\\ &= \int_{S^2}|\nabla u|^2\, dA + 4\int_{S^2} u\circ h_{\lambda,\varsigma}(e^{\psi_{\lambda,\varsigma}} - 1)\, dA + \int_{S^2}|\nabla\psi_{\lambda,\varsigma}|^2\\ &= \int_{S^2}|\nabla u|^2\, dA + 4\int_{S^2} u\, dA - 4\int_{S^2} u\circ h_{\lambda,\varsigma}\, dA + \int_{S^2}|\nabla\psi_{\lambda,\varsigma}|^2.\end{aligned}$$

Here we have employed the fact that $\psi_{\lambda,\varsigma}$ satisfies (cf. [**5**])

$$-\Delta\psi_{\lambda,\varsigma} = 2e^{\psi_{\lambda,\varsigma}} - 2. \tag{3.4}$$

Meanwhile, a direct computation shows

$$\frac{1}{2}\int_{S^2}|\nabla\psi_{\lambda,\varsigma}|^2\, dA + 2\int_{S^2}\psi_{\lambda,\varsigma}\, dA = 0.$$

Substitute this into (3.3) to get

$$\frac{1}{2}\int_{S^2}|\nabla(A_{\lambda,\varsigma}u)|^2\, dA + 2\int_{S^2} A_{\lambda,\varsigma}u\, dA = \int_{S^2}\left(\frac{1}{2}|\nabla u|^2 + 2u\right) dA \tag{3.5}$$

which, in addition to (3.2), implies (3.1).

(3) Denote the dual pairing between $H^1(S^2)$ and its dual space by $\langle\cdot,\cdot\rangle$. Then by the definition of the Gâteaux derivative and (3.1), one has

$$\begin{aligned}\langle I'(A_{\lambda,\varsigma}u), v\rangle &= \lim_{t\to 0}\frac{1}{t}\{I(A_{\lambda,\varsigma}u + tv) - I(A_{\lambda,\varsigma}u)\}\\ &= \lim_{t\to 0}\frac{1}{t}\{I(A_{\lambda,\varsigma}(u + tv\circ h_{\lambda,\varsigma}^{-1})) - I(A_{\lambda,\varsigma}u)\}\\ &= \lim_{t\to 0}\frac{1}{t}\{I(u + tv\circ h_{\lambda,\varsigma}^{-1}) - I(u)\} = \langle I'(u), v\circ h_{\lambda,\varsigma}^{-1}\rangle\end{aligned} \tag{3.6}$$

where $h_{\lambda,\varsigma}^{-1}$ is the inverse of $h_{\lambda,\varsigma}$.

If u is a solution of $(**)$, then $I'(u) = 0$ and $\int_{S^2} e^u = 4\pi$. Consequently, by (3.6) and (3.2), $I'(A_{\lambda,\varsigma}u) = 0$ and $\int_{S^2} e^{A_{\lambda,\varsigma}u} = 4\pi$, which implies $A_{\lambda,\varsigma}u$ is also a solution of $(**)$. This completes our proof.

PROPOSITION 3.2. *Define*

$$J_{\lambda,\varsigma}(u) = \int_{S^2}\left(\tfrac{1}{2}|\nabla u|^2 + 2u\right) - 8\pi\ln\int_{S^2} R\circ h_{\lambda,\varsigma}e^u.$$

Then

$$\forall v\in H^1(S^2),\quad \langle J'_{\lambda,\varsigma}(A_{\lambda,\varsigma}u), v\rangle = \langle J'(u), v\circ h_{\lambda,\varsigma}^{-1}\rangle. \tag{3.7}$$

PROOF.

$$\begin{aligned}\langle J'_{\lambda,\varsigma}(A_{\lambda,\varsigma}u), v\rangle = &\int_{S^2} \nabla(u\circ h_{\lambda,\varsigma})\nabla v + \int_{S^2} \nabla\psi_{\lambda,\varsigma}\cdot\nabla v \\ &+ 2\int_{S^2} v - \frac{8\pi}{\int_{S^2} R\circ h_{\lambda,\varsigma}e^{A_{\lambda,\varsigma}u}}\int_{S^2} R\circ h_{\lambda,\varsigma}e^{A_{\lambda,\varsigma}u}v.\end{aligned}$$

A direct computation leads to

$$\int_{S^2}\nabla(u\circ h_{\lambda,\varsigma})\nabla v = \int_{S^2}\nabla(u\circ h_{\lambda,\varsigma})\nabla(v\circ h^{-1}_{\lambda,\varsigma})\circ h_{\lambda,\varsigma} = \int_{S^2}\nabla u\nabla(v\circ h^{-1}_{\lambda,\varsigma}),$$

$$\int_{S^2} R\circ h_{\lambda,\varsigma}e^{A_{\lambda,\varsigma}u} = \int_{S^2} Re^u,$$

$$\int_{S^2} R\circ h_{\lambda,\varsigma}e^{A_{\lambda,\varsigma}u}v = \int_{S^2} Re^u(v\circ h^{-1}_{\lambda,\varsigma}).$$

By (3.4),

$$\int_{S^2}\nabla\psi_{\lambda,\varsigma}\cdot\nabla v + 2\int_{S^2} v = 2\int_{S^2} e^{\psi_{\lambda,\varsigma}}v = 2\int_{S^2} v\circ h^{-1}_{\lambda,\varsigma}.$$

Therefore

$$\begin{aligned}\langle J'_{\lambda,\varsigma}(A_{\lambda,\varsigma}u), v\rangle &= \int_{S^2}\nabla u\nabla(v\circ h^{-1}_{\lambda,\varsigma}) + 2\int_{S^2} v\circ h^{-1}_{\lambda,\varsigma} \\ &\quad - \frac{8\pi}{\int_{S^2} Re^u}\int_{S^2} Re^u(v\circ h^{-1}_{\lambda,\varsigma}) \\ &= \langle J'(u), v\circ h^{-1}_{\lambda,\varsigma}\rangle.\end{aligned}$$

This completes the proof.

PROPOSITION 3.3. *For any $u\in H_*$, there exist $\lambda\in(0,1]$ and $\varsigma\in S^2$ such that*

$$P(A_{\lambda,\varsigma}u) = 0. \tag{3.8}$$

PROOF. Use the spherical polar coordinate system with pole ς introduced at the beginning of this section, and denote the corresponding orthogonal coordinate system in $\mathbf{R}^3$ by $(x_1, x_1, x_3)_\varsigma$. That is

$$x_1 = r\sin\theta\cos\varphi,\quad x_2 = r\sin\theta\sin\varphi,\quad x_3 = r\cos\theta.$$

In this system, let $P(A_{\lambda,\varsigma}u) = (a_1(\lambda), a_2(\lambda), a_3(\lambda))_\varsigma$. We first show that for any fixed $\varsigma\in S^2$, as $\lambda\to 0$,

$$(a_1(\lambda),\ a_2(\lambda),\ a_3(\lambda))_\varsigma \to (0,0,-1)_\varsigma. \tag{3.9}$$

In fact,

$$a_i(\lambda) = \int_{S^2} x_i\circ h^{-1}_{\lambda,\varsigma}e^u \Big/ \int_{S^2} e^u,\qquad i=1,2,3.$$

It is easily seen that as $\lambda\to 0$

$$\begin{aligned} x_i\circ h^{-1}_{\lambda,\varsigma} &\to 0,\quad \text{in } L^2(S^2),\qquad i=1,2,\\ x_3\circ h^{-1}_{\lambda,\varsigma} &\to -1,\quad \text{in } L^2(S^2).\end{aligned}$$

Hence (3.9) is valid.

Set $F(x) = P(A_{1-t,\varsigma}u)$ with $x = t\varsigma$, $t \in [0,1)$, $\varsigma \in S^2$. Then F is a continuous mapping from $B^3 = [0,1) \times S^2 \to B^3$. (3.9) enables us to extend F continuously to $\overline{B}^3$, the closure of B^3, so that on $\partial B^3 = S^2$, $F(x) = -x$. By a well-known result on topological degree, we have

$$\deg(F, B^3, 0) = \deg(-x, B^3, 0) \neq 0.$$

So there exist $t \in [0,1)$, $\varsigma \in S^2$, such that $F(t\varsigma) = 0$. That is, (3.8) holds. This completes the proof.

PROPOSITION 3.4. *Assume $\{u_k\} \subset H_*$, $P(u_k) \to \bar{\varsigma} \in S^2$. Choose λ_k, ς_k, such that $P(A_{\lambda_k,\varsigma_k}u_k) = 0$. Then $\lambda_k \to 0$ and $\varsigma_k \to \bar{\varsigma}$ as $k \to \infty$.*

PROOF. (1) Suppose there exists a subsequence $\{\lambda_{k_i}\}$ of $\{\lambda_k\}$ such that $\lambda_{k_i} \to \lambda_0 > 0$. Then passing to a subsequence, we have $\lambda_k \to \lambda_0$ and $\varsigma_k \to \varsigma_0$, as $k \to \infty$ for some $\varsigma_0 \in S^2$.

Fix an orthogonal coordinate system $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$. Then $\lambda_0 > 0$ implies that, as $k \to \infty$,

$$x_i \circ h^{-1}_{\lambda_k,\varsigma_k}(x) \to x_i \circ h^{-1}_{\lambda_0,\varsigma_0}(x), \quad \text{uniformly for } x \in S^2,\ i = 1,2,3. \tag{3.10}$$

Let $P(A_{\lambda_k,\varsigma_k}u_k) = (a_1^k, a_2^k, a_3^k)$. Since

$$\int_{S^2} x_i e^{A_{\lambda_k,\varsigma_k}u_k} = \int_{S^2} x_i \circ h^{-1}_{\lambda_k,\varsigma_k} e^{u_k} \quad \text{and} \quad \int_{S^2} e^{A_{\lambda_k,\varsigma_k}u_k} = \int_{S^2} e^{u_k},$$

(3.10) leads to

$$a_i^k = \frac{\int_{S^2} x_i \circ h^{-1}_{\lambda_0,\varsigma_0} e^{u_k}}{\int_{S^2} e^{u_k}} + o(1).$$

Applying Lemma 1.1 to the function $x_i \circ h^{-1}_{\lambda_0,\varsigma_0}$ instead of $R(x)$, we obtain, as $k \to \infty$, $a_i^k \to x_i \circ h^{-1}_{\lambda_0,\varsigma_0}(\bar{\varsigma})$; that is $P(A_{\lambda_k,\varsigma_k}u_k) \to h^{-1}_{\lambda_0,\varsigma_0}(\bar{\varsigma}) \in S^2$, obviously a contradiction with our assumption $P(A_{\lambda_k,\varsigma_k}u_k) = 0$. Hence, one must have

$$\lambda_k \to 0, \quad \text{as } k \to \infty. \tag{3.11}$$

(2) Suppose there exists a subsequence of $\{\varsigma_k\}$ (still denoted by $\{\varsigma_k\}$) such that $\varsigma_k \to \varsigma_0 \neq \bar{\varsigma}$. Choose an orthogonal coordinate system $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$, such that $\varsigma_0 = (0,0,1)$. Then it is easy to see that for $0 < \varepsilon < r(\bar{\varsigma}, \varsigma_0)$, as $k \to \infty$, $x_3 \circ h^{-1}_{\lambda_k,\varsigma_k} \to -1$, uniformly on $S^2 \backslash S_\varepsilon(\varsigma_0)$. Now by a similar argument as in step (1), we arrive at $a_3^k \to -1$, as $k \to \infty$, again a contradiction to $P(A_{\lambda_k,\varsigma_k}u_k) = 0$. This completes the proof.

4. Mass center analysis.

LEMMA 4.1. *All solutions of*

$$-\Delta u + 2 - 2e^u = 0, \qquad x \in S^2, \tag{**}$$

are rotationally symmetric with respect to some axis and hence assume the form

$$\varphi_{\alpha,\varsigma}(x) = \ln \frac{1 - \alpha^2}{(1 - \alpha \cos r(\varsigma, x))^2} \quad \textit{with } \alpha \in [0,1).$$

PROOF. Suppose u is a solution of (**). Then $e^u g_0$ is a metric on S^2 having constant Gaussian curvature 1, where g_0 is the standard metric of S^2. By well-known results in differential geometry, $(S^2, e^u g_0)$ is isometric to (S^2, g_0); i.e., there exists a diffeomorphism φ: $S^2 \to S^2$ such that $\varphi^* g_0 = e^u g_0$. It follows that φ is a conformal transformation of (S^2, g_0). Since all conformal transformations of (S^2, g_0) are explicitly known, we see easily that u has to be rotationally symmetric with respect to some axis.

Moreover, by a result of Hong (cf. [**5**, Lemma 3.1]), u must equal $\varphi_{\alpha,\varsigma}(x)$ with $\alpha \in [0,1)$ and $\varsigma \in S^2$.

PROPOSITION 4.2. *Suppose* $\{u_k\} \subset H_*$, $\{J(u_k)\}$ *bounded*, $J'(u_k) \to 0$ *and* $P(u_k) \to \varsigma \in S^2$, *as* $k \to \infty$. *Then there exists a subsequence* $\{u_{k_i}\}$ *of* $\{u_k\}$ *and corresponding* $\{\alpha_i\}$, $\{\varsigma_i\}$, *with* $\alpha_i \to 1$, $\varsigma_i \to \varsigma$ *as* $i \to \infty$ *such that*

$$\int_{S^2} |\nabla(u_{k_i} - \varphi_{\alpha_i,\varsigma_i})|^2 \to 0, \quad as\ i \to \infty.$$

PROOF. (1) By Proposition 3.3, there exist $\lambda_k, \varsigma_k \in S^2$, such that $P(A_{\lambda_k,\varsigma_k} u_k) = 0$. Let $v_k = A_{\lambda_k,\varsigma_k} u_k$, $\tilde{v}_k = v_k - (1/4\pi)\int_{S^2} v_k$. By Lemma 1.1,

$$J(u_k) = I(u_k) - 8\pi \ln(R(\varsigma) + o(1)) \tag{4.2}$$

which implies the boundedness of $\{I(u_k)\}$. Taking (3.1) into account and by the obvious fact $I(u+C) = I(u)$ for any $u \in H^1(S^2)$ for any constant C, we see that

$$I(\tilde{v}_k) = I(u_k). \tag{4.3}$$

Hence $\{I(\tilde{v}_k)\}$ are bounded. And apparently $P(\tilde{v}_k) = P(v_k) = 0$. Now, due to Proposition 1.2 in [**6**], $\{\tilde{v}_k\}$ is bounded in $H^1(S^2)$, so there exists a subsequence of $\{\tilde{v}_k\}$ (still denoted by $\{\tilde{v}_k\}$) converging weakly to $v_0 \in H^1(S^2)$. Obviously

$$\int_{S^2} v_0 = 0. \tag{4.4}$$

(2) Applying (3.7), we have

$$\begin{aligned}\langle J'_{\lambda_k,\varsigma_k}(\tilde{v}_k), v\rangle &= \langle J'_{\lambda_k,\varsigma_k}(v_k), v\rangle = \langle J'(u_k), v \circ h^{-1}_{\lambda_k,\varsigma_k}\rangle \\ &= \langle J'(u_k), w_k\rangle \quad \forall v \in H^1(S^2)\end{aligned} \tag{4.5}$$

where

$$w_k = v \circ h^{-1}_{\lambda_k,\varsigma_k} - \frac{1}{4\pi}\int_{S^2} v \circ h^{-1}_{\lambda_k,\varsigma_k}.$$

Since $\int_{S^2} w_k = 0$ and $\int_{S^2} |\nabla w_k|^2 = \int_{S^2} |\nabla(v \circ h^{-1}_{\lambda_k,\varsigma_k})|^2 = \int_{S^2} |\nabla v|^2$ we infer from (4.5) that

$$|\langle J'_{\lambda_k,\varsigma_k}(\tilde{v}_k), v\rangle| \le C\|J'(u_k)\| \left(\int_{S^2} |\nabla v|^2\right)^{1/2} \tag{4.6}$$

with some constant C independent of u_k and v.

By Proposition 3.4, $P(u_k) \to \varsigma \in S^2$ implies, as $k \to \infty$, $\varsigma_k \to \varsigma$ and $\lambda_k \to 0$; hence

$$R \circ h_{\lambda_k,\varsigma_k} \to R(\varsigma) \quad \text{in } L^2(S^2). \tag{4.7}$$

And it follows that

$$\langle J'_{\lambda_k,\varsigma_k}(\tilde v_k), v\rangle \to \langle I'(v_0), v\rangle, \quad \text{as } k\to\infty,\ \forall v\in H^1(S^2).$$

On the other hand, by (4.6)

$$\langle J'_{\lambda_k,\varsigma_k}(\tilde v_k), v\rangle \to 0, \quad \text{as } k\to\infty.$$

Therefore,

$$\langle I'(v_0), v\rangle = 0 \quad \forall v\in H^1(S^2). \tag{4.8}$$

By Lemma 4.1, $v_0(x)=\varphi_{\alpha,\eta}(x)+C$, for some constant C, α and $\eta\in S^2$. The weak convergence of $\{\tilde v_k\}$ to v_0 and $P(\tilde v_k)=0$ imply $P(v_0)=0$, and it follow that $\alpha=0$ and $v_0=\text{const}$. Now, by (4.4), we obtain $v_0=0$.

(3) Due to (4.6), $\|J'(u_k)\|\to 0$ and $\tilde v_k \rightharpoonup 0$, we have

$$o(1)=\langle J'_{\lambda_k,\varsigma_k}(\tilde v_k),\tilde v_k\rangle = \int_{S^2}|\nabla\tilde v_k|^2 - 8\pi\frac{\int_{S^2} R\circ h_{\lambda_k,\varsigma_k}e^{\tilde v_k}\tilde v_k}{\int_{S^2} R\circ h_{\lambda_k,\varsigma_k}e^{\tilde v_k}}.$$

It is not difficult to see from (4.7) that the last term in the above equality vanishes as $k\to\infty$ since the boundedness of $\{J(u_k)\}$ and $P(u_k)\to\varsigma$ imply $R(\varsigma)>0$ (cf. [**6**]). Hence

$$\int_{S^2}|\nabla\tilde v_k|^2\to 0, \quad \text{as } k\to\infty, \tag{4.9}$$

which means $\tilde v_k\to 0$ strongly in $H^1(S^2)$.

(4) By (4.9),

$$\int_{S^2}|\nabla(u\circ h_{\lambda_k,\varsigma_k}+\psi_{\lambda,\varsigma_k})|^2\to 0.$$

It follows that $\int_{S^2}|\nabla(u_k+\psi_{\lambda_k,\varsigma_k}\circ h^{-1}_{\lambda_k,\varsigma_k})|^2\to 0$, due to the conformal invariance of the integral $\int_{S^2}|\nabla u|^2$. Let $\alpha_k=(1-\lambda_k^2)/(1+\lambda_k^2)$. Then a straightforward computation shows

$$\psi_{\lambda_k,\varsigma_k}\circ h^{-1}_{\lambda_k,\varsigma_k}(x) = -\varphi_{\alpha_k,\varsigma_k}(x).$$

Obviously $\alpha_k\to 1$, since $\lambda_k\to 0$. This completes our proof.

PROPOSITION 4.3. *Let $\{u_k\}\subset H_*$ and assume $\{J(u_k)\}$ is bounded, $J'(u_k)\to 0$, and $P(u_k)\to\varsigma\in S^2$ as $k\to\infty$. Then there is a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ such that $J(u_{k_i})\to -8\pi\ln 4\pi R(\varsigma)$.*

PROOF. By (4.2), (4.3) and (4.9), one can pick a subsequence $\{u_{k_i}\}$ of u_k such that

$$\begin{aligned} J(u_{k_i}) &= I(u_{k_i}) - 8\pi\ln(R(\varsigma)+o(1)) = I(\tilde v_{k_i}) - 8\pi\ln(R(\varsigma)+o(1))\\ &\to I(0)-8\pi\ln R(\varsigma) = -8\pi\ln 4\pi R(\varsigma).\end{aligned}$$

This completes the proof.

PROPOSITION 4.4. *Assume $\{u_k\}$, $\{v_k\}\subset H_*$.*

(1) $\{J(u_k)\}$, $\{J(v_k)\}$ bounded;

(2) $\int_{S^2}|\nabla(u_k-v_k)|^2\to 0$, as $k\to\infty$;

(3) $P(u_k)\to\eta\in S^2$, $P(v_k)\to\varsigma\in S^2$, as $k\to\infty$.

Then $\eta = \varsigma$.

PROOF. We argue indirectly. Suppose $\eta \neq \varsigma$.

By Proposition 4.2, one can pick a subsequence of $\{v_k\}$ (still denoted by $\{v_k\}$) and corresponding $\{\alpha_k\}$, $\{\varsigma_k\}$ with $\alpha_k \to 1$, $\varsigma_k \to \varsigma$ such that

$$\int_{S^2} |\nabla(v_k - \varphi_{\alpha_k,\varsigma_k})|^2 \to 0, \quad \text{as } k \to \infty.$$

From assumption (2)

$$\int_{S^2} |\nabla(\tilde{u}_k - \tilde{\varphi}_{\alpha_k,\varsigma_k})|^2 \to 0, \quad \text{as } k \to \infty. \tag{4.10}$$

Here we again used the notation $\tilde{u} = u - (1/4\pi)\int_{S^2} u$.

Noting that $P(\tilde{u}) = P(u)\ \forall u \in H^1(S^2)$, $\int_{S^2} |\nabla\tilde{\varphi}_{\alpha_k,\varsigma_k}|^2 \to \infty$ and $I(\tilde{\varphi}_{\alpha_k,\varsigma_k}) = -8\pi\ln 4\pi$; applying Lemma 2.2 in [**5**], we obtain, for any $x \in S^2$, $x \neq \varsigma$,

$$\tilde{\varphi}_{\alpha_k,\varsigma_k}(x) \to -\infty, \quad \text{as } k \to \infty. \tag{4.11}$$

Let $\varepsilon = \frac{1}{2}r(\eta,\varsigma)$, $S_k = \{x \in S_\varepsilon(\eta) \colon \tilde{u}_k(x) \geq 0\}$, $\tilde{u}_k^+ = \max\{\tilde{u}_k, 0\}$; then by (4.10),

$$\int_{S^k} \{|\nabla(\tilde{u}_k^+ - \tilde{\varphi}_{\alpha_k,\varsigma_k})|^2 + |\tilde{u}_k^+ - \tilde{\varphi}_{\alpha_k,\varsigma_k}|^2\} \to 0, \quad \text{as } k \to \infty.$$

Now, the boundedness of $\int_{S_k} |\nabla\tilde{\varphi}_{\alpha_k,\varsigma_k}|^2$ and (4.11) imply the boundedness of $\int_{S_k}(|\nabla\tilde{u}_k^+|^2 + \tilde{u}_k^{+2})$, hence of $\int_{S_\varepsilon(\eta)}(|\nabla\tilde{u}_k^+|^2 + |\tilde{u}_k^+|^2)$. Consequently, by Theorem 2.46 in [**8**], we infer that $\int_{S_\varepsilon(\eta)} e^{\tilde{u}_k^+}$ are bounded. It follows, from $P(u_k) \to \eta \in S^2$ and the proof of Lemma 1.1, that $\int_{S^2} e^{\tilde{u}_k}$ are bounded. Assumption (1) implies $\int_{S^2} |\nabla\tilde{u}_k|^2$ are bounded; hence there exists a subsequence $\{\tilde{u}_{k_i}\}$ of $\{\tilde{u}_k\}$ converging weakly to some element $u_0 \in H^1(S^2)$, which leads to $P(\tilde{u}_{k_i}) \to P(u_0)$. However, it is evidently that $P(u_0)$ can never lie on S^2. This contradicts $\eta \in S^2$, and the proof is completed.

LEMMA 4.5. *If* λ *is sufficiently close to* 1, *then*

$$\mu_\lambda < -8\pi\ln 4\pi\nu. \tag{4.12}$$

Here we assume $\nu > 0$.

PROOF. Using h_0 in (R_2), set

$$l_\lambda(t) = \varphi_{\lambda,h_0(t)}, \qquad t \in [0,1].$$

Apparently, for λ sufficiently close to 1, $l_\lambda \in L_\lambda$. We want to show that

$$\max_{l_\lambda([0,1])} J < -8\pi\ln 4\pi\nu.$$

Since $J(\varphi_{\lambda,\varsigma}) = -8\pi\ln\int_{S^2} Re^{\varphi_{\lambda,\varsigma}}$, it suffices to verify the inequality

$$\int_{S^2} Re^{\varphi_{\lambda,\varsigma}} > 4\pi\nu \quad \forall\varsigma \in h_0([0,1]). \tag{4.13}$$

Set $N_\delta = \{x \in h_0([0,1]) \colon \operatorname{dist}(x,K) < \delta\}$, where K was defined in (R_2) and $\operatorname{dist}(\cdot,\cdot)$ stands for the geodesic distance on standard S^2. By (R_2), one can choose a small δ so that

$$\Delta R|_{N_\delta} > 0. \tag{4.14}$$

(1) $\varsigma \in h_0([0,1])\backslash N_\delta$. In this case, there is an $\varepsilon > 0$, such that

$$R(\varsigma) \geq \nu + \varepsilon \quad \forall \varsigma \in h_0([0,1])\backslash N_\delta.$$

Let λ be sufficiently close to 1 so that $d(\varphi_{\lambda,\varsigma}) \leq (\varepsilon/2C_0)^3$. (Note that here $d(\varphi_{\lambda,\varsigma})$ is independent of ς.) Then by Lemma 1.1, we have

$$\int_{S^2} Re^{\varphi_{\lambda,\varsigma}} \geq (R(\varsigma) - \varepsilon/2) \int_{S^2} e^{\varphi_{\lambda,\varsigma}} = 4\pi(R(\varsigma) - \varepsilon/2) > 4\pi\nu.$$

(2) $\varsigma \in N_\delta$. Due to the continuity of J and the compactness of $\overline{N}_\delta$, the closure of N_δ on S^2, it suffices to show that for each $\varsigma \in N_\delta$ there is a $\lambda(\varsigma)$ such that, as $1 > \lambda \geq \lambda(\varsigma)$, (4.13) holds. For this aim, again choose a spherical polar coordinate system $x(\theta, \psi)$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$, so that $\varsigma = (0, \psi)$. Then for any $\varepsilon > 0$,

$$\begin{aligned}\int_{S^2} Re^{\varphi_{\lambda,\varsigma}} &= (1-\lambda^2)\left\{\int_0^{2\pi}\int_0^{\varepsilon} \frac{[R(x) - R(\varsigma)]\sin\theta}{(1-\lambda\cos\theta)^2}\,d\theta\,d\psi + \int_0^{2\pi}\int_\varepsilon^{\pi}(\cdots)\right\} \\ &\quad + 4\pi R(\varsigma) \\ &\equiv (1-\lambda^2)\{(\mathrm{I}) + (\mathrm{II})\} + 4\pi R(\varsigma) \\ &\geq (1-\lambda^2)\{(\mathrm{I}) + (\mathrm{II})\} + 4\pi\nu.\end{aligned}$$

Thus one need only verify that

$$(\mathrm{I}) + (\mathrm{II}) > 0 \text{ as } \lambda \text{ sufficiently close to 1.} \tag{4.15}$$

In fact, for any fixed ε, integral (II) is bounded for all $\lambda \leq 1$. Using the second order Taylor expansion of R at point ς, taking into account that $\varphi_{\lambda,\varsigma}$ depends on θ only, and by a direct calculation, we arrive at

$$(\mathrm{I}) = \pi\int_0^{\varepsilon} \frac{\{\Delta R(\varsigma)\sin^2\theta + o(\theta^2)\}}{(1-\lambda\cos\theta)^2}\sin\theta\,d\theta.$$

Let ε be so small that

$$(\mathrm{I}) \geq \frac{\pi}{2}\Delta R(\varsigma)\int_0^{\varepsilon}\frac{\sin^3\theta\,d\theta}{(1-\lambda\cos\theta)^2}.$$

Then an integration by parts shows

$$\int_0^{\varepsilon}\frac{\sin^3\theta\,d\theta}{(1-\lambda\cos\theta)^2} \to +\infty, \quad \text{as } \lambda \to 1.$$

Therefore (4.15) holds, and the proof is completed.

Let λ_0 be so close to 1 that both (1.5) and (4.12) hold. We write $\mu = \mu_{\lambda_0}$ and $L = L_{\lambda_0}$.

PROPOSITION 4.6. *There exist α_0, $\delta_0 > 0$, such that for any $\{v_k\}$ in H_*, if $J(v_k) \leq \mu + \delta_0$ $(k = 1, 2, \dots)$ and $P(v_k) \to \varsigma \in S^2$, as $k \to \infty$; then*

$$R(\varsigma) \geq \nu + \alpha_0. \tag{4.16}$$

PROOF. Estimate (4.12) implies the existence of constants α_0, $\delta_0 > 0$, such that

$$\mu + \delta_0 \leq -8\pi\ln 4\pi(\nu + \alpha_0). \tag{4.17}$$

By (1.2) and (1.4)

$$J(v_k) \geq -8\pi\ln 4\pi[R(Q(v_k)) + C_0\sqrt[3]{d(v_k)}].$$

Clearly, as $k \to \infty$, both $d(v_k) \to 0$ and $R(Q(v_k)) \to R(\varsigma)$; hence

$$\mu + \delta_0 \geq \overline{\lim_k} J(v_k) \geq -8\pi \ln 4\pi R(\varsigma),$$

which, with (4.17), implies $R(\varsigma) \geq \nu + \alpha_0$. This completes the proof.

5. Constructing a continuous "flow" in H_* to complete the proof of the Theorem. Let $M = R^{-1}(m) \equiv \{x \in S^2 \colon R(x) = m\}$. Choose $\varepsilon_1 > 0$, so that

$$\mu > -8\pi \ln 4\pi (m - \varepsilon_1). \tag{5.1}$$

Define $U_d = \{u \in H_* \colon R(Q(u)) \leq m - \varepsilon_1, d(u) \leq d\}$. Let $\mu_0 = \inf_{H_*} J = -8\pi \ln 4\pi m$.

LEMMA 5.1 (*A result of a continuous "flow"*). *There exist $\delta, d > 0$ and a continuous mapping $\mathcal{T} \colon H_* \to H_*$, such that*

(a) $J(\mathcal{T}(u)) \leq J(u)$, $\forall u \in H_*$;
(b) $\mathcal{T}(J^{-1}(\mu_0, \mu + \delta) \cap U_d) \subset J^{-1}(\mu_0, \mu - \delta)$;
(c) $\mathcal{T}|_{J^{-1}(\mu_0, \mu_0 + \delta)} = \mathrm{id}$, $\mu_0 + \delta < \mu - \delta$;
(d) $\mathcal{T}(H_* \backslash U_d) \subset H_* \backslash U_d$,

where $J^{-1}(\alpha, \beta)$ stands for $\{u \in H_* \colon \alpha < J(u) < \beta\}$.

PROOF. Analogous to the proof of Proposition 4.6 and by the definition of U_d, it is easy to show that there are constants $d_1, \delta_1, \varepsilon_2 > 0$, such that if $\delta < \delta_1$, $d < d_1$,

$$\forall u \in J^{-1}(\mu_0, \mu + \delta) \cap U_d, \qquad Q(u) \in R^{-1}[\nu + \varepsilon_2, m - \varepsilon_1]. \tag{5.2}$$

By (R_3), there is a constant $\alpha_1 > 0$, such that

$$|\nabla R(x)| \geq \alpha_1, \quad \forall x \in R^{-1}[\nu + \varepsilon_2/2, m - \varepsilon_1/2]. \tag{5.3}$$

(1) Let $u \in H_*$ such that $Q(u) \in R^{-1}[\nu + \varepsilon_2/2, m - \varepsilon_1/2]$. Let $z(u)$ be the straight line passing through 0 which is perpendicular to the plane spanned by the vectors $Q(u)$ and $\nabla R(Q(u))$. Define $T(\theta, u)$ to be the rotation in $\mathbf{R}^3$ which takes $z(u)$ as its axis and which rotates along the direction $\nabla R(Q(u))$ by angle θ.

Let u be fixed for a moment, and write $x_0 = Q(u)$, $T_\theta = T(\theta, u)$. Consider

$$f(\theta) = \left(\int_{S^2} e^u\right)^{-1} \left(\int_{S^2} R(x) e^{u(T_\theta^{-1} x)} - \int_{S^2} R(x) e^{u(x)}\right)$$

where T_θ^{-1} is the inverse of T_θ. Noting that T_θ is an orthogonal transformation in $\mathbf{R}^3$, we arrive immediately at

$$f(\theta) = \left(\int_{S^2} e^u\right)^{-1} \cdot \int_{S^2} [R(T_\theta x) - R(x)] e^{u(x)}.$$

The first order Taylor expansion of R at x_0 leads to

$$R(T_\theta x_0) - R(x_0) \geq \tfrac{1}{2} |\nabla R(x_0)| \cdot |T_\theta x_0 - x_0|, \tag{5.4}$$

for θ sufficiently small. Let

$$\theta(x_0) = \max\{\alpha \colon \text{as } \theta \leq \alpha, (5.4) \text{ is valid}\},$$
$$\theta_0 = \inf\{\theta(x_0) \colon x_0 \in R^{-1}[\nu + \varepsilon_2/2, m - \varepsilon_1/2]\}.$$

Then by the smoothness of R and the compactness of $R^{-1}[\nu+\varepsilon_2/2, m-\varepsilon_1/2]$, we have $\theta_0 > 0$. (5.4) and the continuity of R imply, as r sufficiently small,

$$R(T_{\theta_0}x) - R(x) \geq \tfrac{1}{4}|\nabla R(x_0)|\,|T_{\theta_0}x_0 - x_0| \quad \forall x \in S_r(x_0). \tag{5.5}$$

Let $r(x_0) = \max\{s\colon \text{as } r \leq s,\ (5.5) \text{ holds}\}$. Define

$$r_0 = \inf\{r(x_0)\colon x_0 \in R^{-1}[\nu+\varepsilon_2/2, m-\varepsilon_1/2]\}.$$

Then similarly, one has $r_0 > 0$.

From the proof of Lemma 1.1, we see that there exists $d_2 > 0$ such that if $d(u) \leq d_2$, then

$$\frac{\int_{S_{r_0}(x_0)} e^u}{\int_{S^2} e^u} \geq \frac{1}{2} \quad \text{and} \quad \max_{S^2}|\nabla R|\frac{\int_{S^2\setminus S_{r_0}(x_0)} e^u}{\int_{S^2} e^u} \leq \frac{\alpha_1}{16}. \tag{5.6}$$

Apparently, d_2 is independent of x_0.

Now, by (5.3), (5.5) and (5.6), noting that for any θ, x, $|T_\theta x_0 - x_0| \geq |T_\theta x - x|$, we obtain, for all $\theta \leq \theta_0$,

$$\begin{aligned} f(\theta) &= \left(\int_{S^2} e^u\right)^{-1}\left\{\int_{S_{r_0}(x_0)}[R(T_\theta x) - R(x)]e^u + \int_{S^2\setminus S_{r_0}(x_0)}(\cdots)\right\} \\ &\geq \frac{1}{4}|\nabla R(x_0)|\cdot|T_\theta x_0 - x_0|\frac{\int_{S_{r_0}(x_0)} e^u}{\int_{S^2} e^u} \\ &\quad - \max_{S^2}|\nabla R|\frac{\int_{S^2\setminus S_{r_0}(x_0)}|T_\theta x - x|e^u}{\int_{S^2} e^u} \\ &\geq \frac{\alpha_1}{16}|T_\theta x_0 - x_0| \equiv \alpha_2|T_\theta x_0 - x_0|. \end{aligned} \tag{5.7}$$

(2) Choose $\delta_2, d_3 > 0$ so small that

$$m\cdot e^{-\delta_2/8\pi} - C_0 d_3^{1/3} > m - \varepsilon_1/2$$

where C_0 was defined in Lemma 1.1. Then by the inequality

$$J(u) \geq -8\pi \ln 4\pi[R(Q(u)) + C_0\sqrt[3]{d(u)}]$$

derived from (1.2) and (1.4), it is easy to verify that for all $u \in H_*$

$$\text{if } J(u) \leq \mu_0 + \delta_2 \text{ and } d(u) \leq d_3, \text{ then } R(Q(u)) > m - \varepsilon_1/2. \tag{5.8}$$

Let

$$G = \left\{x \in B^3\colon \frac{x}{|x|} \in R^{-1}[\nu+\varepsilon_2/2, m-\varepsilon_1/2], 1-|x| \leq \min\{d_2, d_3\}\right\},$$

$$G' = \left\{x \in G\colon \frac{x}{|x|} \in R^{-1}[\nu+\varepsilon_2, m-\varepsilon_1], 1-|x| \leq \frac{1}{2}\min\{d_2, d_3\}\right\}.$$

Choose a C^∞ function g on $\overline{B}^3$, satisfying $0 \leq g(x) \leq 1$, $\forall x \in \overline{B}^3$; $g \equiv 1$, $x \in G'$; $g \equiv 0$, $x \in \overline{B}^3\setminus G$. Define

$$\mathcal{T}_t u(x) = u(T^{-1}(\tan(P(u)), u)x), \qquad t \in [0, \theta_0].$$

Note that $P(u)$, $Q(u)$ depend continuously on u (in the $H^1(S^2)$ topology) and R is smooth. We see that $T^{-1}(\theta, u)$ depends continuously on u, while the continuity of $T^{-1}(\theta, u)$ with respect to θ is obvious. Therefore $\mathcal{T}$ is a continuous map from $[0, \theta_0] \times H^1(S^2)$ to $H^1(S^2)$, and thus defines a continuous "flow" in $H^1(S^2)$ which is nonincreasing with respect to the functional J, that is

$$J(\mathcal{T}_t u) \leq J(u) \quad \forall u \in H_*, t \in [0, \theta_0]. \tag{5.9}$$

In case $P(u) \notin G$, the above inequality holds obviously, since $g(P(u)) = 0$, $\mathcal{T}_t u = u$. And in case $P(u) \in G$, by (5.7)

$$\int_{S^2} R(x) e^{\mathcal{T}_t u} \geq \int_{S^2} R(x) e^u \quad \forall t \in [0, \theta_0].$$

Moreover, since $T^{-1}(\theta, u)$ is an orthogonal transformation we have

$$\frac{1}{2} \int_{S^2} |\nabla(\mathcal{T}_t u)|^2 + 2 \int_{S^2} \mathcal{T}_t u = \frac{1}{2} \int_{S^2} |\nabla u|^2 + 2 \int_{S^2} u,$$

for any $t \in [0, \theta_0]$, $u \in H^1(S^2)$. Therefore (5.9) also holds.

If $P(u) \in G'$, by the definition of g

$$\mathcal{T}_t u(x) = u(T^{-1}(t, u)x).$$

Write J as

$$J(u) = I(u) - 8\pi \ln \frac{\int_{S^2} R e^u}{\int_{S^2} e^u}.$$

It is easily see that $\int_{S^2} e^{\mathcal{T}_t u} = \int_{S^2} e^u$; hence $I(\mathcal{T}_t u) = I(u)$. By (5.7)

$$\frac{\int_{S^2} R(x) e^{\mathcal{T}_{\theta_0} u}}{\int_{S^2} e^{\mathcal{T}_{\theta_0} u}} - \frac{\int_{S^2} R(x) e^u}{\int_{S^2} e^u} \geq \alpha_2 |T(\theta_0, u) Q(u) - Q(u)| \geq \alpha_3 > 0.$$

Consequently, there exists $\delta_3 > 0$ such that

$$J(\mathcal{T}_{\theta_0} u) \leq J(u) - \delta_3, \quad \text{for all } u \in H_*, \ P(u) \in G'. \tag{5.10}$$

(3) Now define $\mathcal{T}(u) = \mathcal{T}_{\theta_0} u$. Let

$$\delta < \min\{\delta_1, \delta_2, \delta_3/2\}, \qquad d < \min\{d_1, d_2/2, d_3/2\}.$$

Then equation (5.9) implies (a). (5.10) implies (b), since by (5.2), for any $u \in J^{-1}(\mu_0, \mu + \delta) \cap U_d$, $P(u) \in G'$. (5.8) and the definition of G and of g imply (c). Finally, noting that $d(\mathcal{T}_t u) = d(u)$, $R(Q(\mathcal{T}_t u)) \geq R(Q(u))$ and by the definition of U_d, we see that the conclusion (d) of the lemma is true. This completes the proof.

PROOF OF THE THEOREM. For simplicity, we write U_d in Lemma 5.1 by U, and $l([0,1])$ by l, for $l \in L$.

Choose $l_k \in L$, $k = 1, 2, \ldots$, such that $\max_{l_k} J(u) < \mu + \delta$ and $\max_{l_k} J \to \mu$, as $k \to \infty$. By (a) and (c) in Lemma 5.1,

$$\mathcal{T}(l_k) \equiv \tilde{l}_k \in L \quad \text{and} \quad \max_{\tilde{l}_k} J \to \mu. \tag{5.11}$$

And by (b) and (d)

$$J|_{\tilde{l}_k \cap U} < \mu - \delta. \tag{5.12}$$

Now choose $u_k \in \tilde{l}_k$, so that $J(u_k) = \max_{\tilde{l}_k} J \to \mu$. It can be shown that (cf. e.g. [**7**], the proof for Mountain Pass Lemma by using Ekeland's variational principle) there exist $\{v_k\} \subset H_*$, such that

$$\|u_k - v_k\|_{H^1} \to 0, \quad J(v_k) \to \mu \quad \text{and} \quad J'(v_k) \to 0. \tag{5.13}$$

By (5.12), $u_k \in H_* \backslash U$. Thus there are only two possibilities:
(1) $d(u_k) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$, or
(2) $P(u_k) \to \varsigma \in S^2$, with $R(\varsigma) > m - \varepsilon_1$.

In case (1), by (5.13), $\{P(v_k)\}$ is bounded away from the sphere S^2. Then by Proposition 2.1 and (5.13), $\{\tilde{v}_k \equiv v_k - (1/4\pi)\int_{S^2} v_k\}$ converges strongly in H_* to some v_0, that $J'(v_0) = 0$ and $J(v_0) = \mu$. Hence μ is a critical value of J.

In case (2), by Proposition 4.4, $P(v_k) \to \varsigma$; then due to Proposition 4.3,

$$J(v_k) \to -8\pi \ln 4\pi R(\varsigma),$$

that is,

$$\mu = -8\pi \ln 4\pi R(\varsigma).$$

By (5.1), $-8\pi\ln 4\pi R(\varsigma) < -8\pi \ln 4\pi(m - \varepsilon_1) < \mu$, a contradiction. Therefore, μ is a critical value of J. This completes the proof of our Theorem.

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